Examples for Chapter 10.

EXAMPLE 10.1

It is desired to find the frequency and power of a single real sinusoidal signal in white noise. The correlation matrix for the data is

$$\mathbf{R}_{\boldsymbol{x}} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

Note that since the real sinusoid consists of two complex exponentials at positive and negative frequencies, M is equal to 2 for this problem and the Pisarenko method thus requires a 3×3 correlation matrix.

The matrices of eigenvalues and eigenvectors of $\mathbf{R}_{\boldsymbol{x}}$ are found to be

$$\boldsymbol{\Lambda} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(You can check this by direct multiplication.)

The noise variance is therefore

$$\sigma_{\rm o}^2 = \lambda_3 = 1$$

and the noise eigenvector is

$$\mathbf{e}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The Pisarenko pseudospectrum is

$$\hat{P}_P(e^{\jmath\omega}) = \frac{1}{|\mathbf{w}^{*T}\mathbf{e}_3|^2}$$

A plot of this function peaks at $\omega = \pm\pi/2$ because for these values

$$\mathbf{w} = \begin{bmatrix} 1 \\ e^{\pm\jmath\frac{\pi}{2}} \\ e^{\pm\jmath\frac{\pi}{2}2} \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\jmath \\ -1 \end{bmatrix}$$

and

$$\mathbf{w}^{*T}\mathbf{e}_3 = \begin{bmatrix} 1 & \mp\jmath & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

The frequencies can also be located by finding the roots of the eigenfilter

$$E_3(z) = -\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} z^{-2}$$

The roots are

$$z = \pm \jmath = e^{\pm \jmath \frac{\pi}{2}}$$

which shows that $\omega = \pm\pi/2$.

To find the power, proceed as follows. The signal vectors are

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ e^{\jmath\frac{\pi}{2}} \\ e^{\jmath\pi} \end{bmatrix} = \begin{bmatrix} 1 \\ \jmath \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_2 = \begin{bmatrix} 1 \\ e^{-\jmath\frac{\pi}{2}} \\ e^{-\jmath\pi} \end{bmatrix} = \begin{bmatrix} 1 \\ -\jmath \\ -1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} - & \mathbf{e}_1^{*T} & - \\ - & \mathbf{e}_2^{*T} & - \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{s}_1 & \mathbf{s}_2 \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \jmath & -\jmath \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -\sqrt{2} \\ -\jmath & \jmath \end{bmatrix}$$

The equation

$$\begin{bmatrix} |\beta_{11}|^2 & |\beta_{12}|^2 \\ |\beta_{21}|^2 & |\beta_{22}|^2 \end{bmatrix} \begin{bmatrix} \mathsf{P}_1 \\ \mathsf{P}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_{\mathrm{o}}^2 \\ \lambda_2 - \sigma_{\mathrm{o}}^2 \end{bmatrix}$$

becomes

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathsf{P}_1 \\ \mathsf{P}_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

which has the solution

$$\mathsf{P}_1 = \mathsf{P}_2 = 1$$

□

EXAMPLE 10.2

The correlation matrix corresponding to complex exponentials in white noise is given by

$$\mathbf{R}_{\boldsymbol{x}} = \begin{bmatrix} 2 & -j & -1 \\ j & 2 & -j \\ -1 & j & 2 \end{bmatrix}$$

The matrices of eigenvalues and eigenvectors are found to be

$$\mathbf{\Lambda} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} -\frac{1}{\sqrt{3}}j & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}j & \frac{1}{\sqrt{2}}j \\ \frac{1}{\sqrt{3}}j & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since the two smallest eigenvalues are identical, the noise subspace has dimension 2, and there is only a single complex exponential present.

The matrix of noise subspace eigenvectors is

$$\mathbf{E}_{noise} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}}\jmath & \frac{1}{\sqrt{2}}\jmath \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and the corresponding projection matrix for the noise subspace is

$$\boldsymbol{P}_{noise} = \mathbf{E}_{noise}\mathbf{E}_{noise}^{*T} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3}\jmath & \frac{1}{3} \\ -\frac{1}{3}\jmath & \frac{2}{3} & \frac{1}{3}\jmath \\ \frac{1}{3} & -\frac{1}{3}\jmath & \frac{2}{3} \end{bmatrix}$$

The MUSIC pseudospectrum is given by

$$\hat{P}_{MU}(e^{\jmath\omega}) = \frac{1}{\mathbf{w}^{*T}\boldsymbol{P}_{noise}\mathbf{w}}$$

It can be verified that the denominator goes to zero for

$$\mathbf{w} = \begin{bmatrix} 1 \\ e^{\jmath\frac{\pi}{2}} \\ e^{\jmath 2\cdot\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ \jmath \\ -1 \end{bmatrix}$$

Therefore the signal has frequency

$$\omega = \tfrac{\pi}{2}$$

To find the frequency by the root MUSIC procedure, it is necessary to form the denominator polynomial

$$\hat{P}_{MU}^{-1}(z) = \sum_{i=M+1}^{N} E_i(z) E_i^*(1/z^*)$$

Using the noise eigenvectors, the eigenfilters are

$$E_2(z) = \sqrt{\tfrac{2}{3}} - \tfrac{1}{\sqrt{6}} \jmath z^{-1} + \tfrac{1}{\sqrt{6}} z^{-2}$$

with

$$E_2(z) E_2^*(1/z^*) = \tfrac{1}{3} z^2 + \tfrac{1}{6} \jmath z + 1 - \tfrac{1}{6} \jmath z^{-1} + \tfrac{1}{3} z^{-2}$$

and

$$E_3(z) = \tfrac{1}{\sqrt{2}} \jmath z^{-1} + \tfrac{1}{\sqrt{2}} z^{-2}$$

with

$$E_3(z) E_3^*(1/z^*) = \tfrac{1}{2} \jmath z + 1 - \tfrac{1}{2} \jmath z^{-1}$$

The required polynomial is therefore

$$\begin{aligned} E_2(z) E_2^*(1/z^*) \;&+\; E_3(z) E_3^*(1/z^*) \\ &= \tfrac{1}{3} z^2 + \tfrac{2}{3} \jmath z + 2 - \tfrac{2}{3} \jmath z^{-1} + \tfrac{1}{3} z^{-2} \end{aligned}$$

This polynomial has a *double* root on the unit circle at $z = \jmath$ corresponding to the frequency $\omega = \pi/2$ and two other roots at $z = -0.2679\jmath$ and $z = -3.7321\jmath$. These last two are the spurious roots. $\square$

EXAMPLE 10.3

The correlation matrix corresponding to the data in Example 10.2 is

$$\mathbf{R}_{\boldsymbol{x}} = \begin{bmatrix} 2 & -\jmath & -1 \\ \jmath & 2 & -\jmath \\ -1 & \jmath & 2 \end{bmatrix}$$

Following the procedure in Example 10.2, the matrix of noise subspace eigenvectors is found to be

$$\mathbf{E}_{noise} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}}\jmath & \frac{1}{\sqrt{2}}\jmath \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The partitioning of this matrix defines the quantities

$$\mathbf{c}^T = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \end{bmatrix}$$

and

$$\mathbf{E}'_{noise} = \begin{bmatrix} -\frac{1}{\sqrt{6}}\jmath & \frac{1}{\sqrt{2}}\jmath \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The terms needed for the vector $\mathbf{d}$ are computed as

$$\mathbf{E}'_{noise}\mathbf{c}/(\mathbf{c}^T\mathbf{c}) = \begin{bmatrix} -\frac{1}{\sqrt{6}}\jmath & \frac{1}{\sqrt{2}}\jmath \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} \\ 0 \end{bmatrix} \Big/ \left(\sqrt{\tfrac{2}{3}}\right)^2$$

$$= \begin{bmatrix} -\frac{1}{3}\jmath \\ \frac{1}{3} \end{bmatrix} \Big/ \frac{2}{3} = \begin{bmatrix} -\frac{1}{2}\jmath \\ \frac{1}{2} \end{bmatrix}$$

Therefore $\mathbf{d}$ is given by

$$\mathbf{d} = \begin{bmatrix} 1 \\ \\ \mathbf{E}'_{noise}\mathbf{c}/(\mathbf{c}^{*T}\mathbf{c}) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2}\jmath \\ \frac{1}{2} \end{bmatrix}$$

The pseudospectrum is then

$$\hat{P}_{MN}(e^{\jmath\omega}) \stackrel{\text{def}}{=} \frac{1}{|\mathbf{w}^{*T}\mathbf{d}|^2}$$

The product $\mathbf{w}^{*T}\mathbf{d}$ goes to zero for

$$\mathbf{w} = \begin{bmatrix} 1 \\ e^{\jmath\frac{\pi}{2}} \\ e^{\jmath 2\cdot\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ \jmath \\ -1 \end{bmatrix}$$

as in the previous example. Therefore the frequency of the complex exponential is found to be

$$\omega = \frac{\pi}{2}$$

as before.

The frequency of the complex exponential can be found be the alternative method of forming the polynomial

$$D(z) = 1 - \tfrac{1}{2}\jmath z^{-1} + \tfrac{1}{2}z^{-2}$$

and finding its roots. These roots are found to be at $z = \jmath$ and $z = -\frac{1}{2}\jmath$. The former is on the unit circle and corresponds to the frequency $\omega = \pi/2$ while the latter, which is a spurious root, lies within the unit circle (as guaranteed for this method). $\square$